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ON THE SEPARATION OF POINTS OF A CONTINUOUS CURVE BY ARCS AND SIMPLE CLOSED CURVES[1]

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It has been shown by R. L. Moore[2] that in order that a plane continuum M be a continuous curve it is necessary and sufficient that for every two points, A and B of M, there should exist a subset K of M consisting of a finite number of subcontinua of M and separating A and B in M. A subset K of M is said to *separate* A *and* B *in* M if M-K is the sum of two non-vacuous mutually separated sets containing A and B, respectively. In this paper it will be shown that if M is a bounded[3] plane continuous curve, the points A and B can be separated by a set of n arcs[4] of M, where n is the number of components[5] of M-A-B that have both A and B as limit points. To do this it will be shown that there exists a simple closed curve which separates A and B in the plane and which has in common with M only a set of n arcs. We shall make considerable use of the properties of the arc-curves which I have developed in another paper.[6] If K is a subset of a continuous curve M the *arc-curve of* K *with respect to* M (denoted by $M(K)$) is the point set consisting of all points $[P]$ such that P lies on some arc[4] of M whose end-points belong to K. It follows that if a subset of M separates A and B in the arc-curve of $A + B$ with respect to M, then this subset separates A and B in M. All the point sets considered in this paper are assumed to lie in a two-dimensional Euclidean space E_2.

The theorem does not hold true if A and B are merely closed mutually exclusive subsets of M as may be seen by the following simple example.[7] Let M consist of the circle with center $(0, 0)$ and radius 3 together with the intervals from $(-3, 0)$ to $(3, 0)$, from $(-2, 0)$ to $(0, -2)$ and from $(0, -1)$ to $(0, -3)$. Let the set A consist of the point $(0, -3)$. Let the set B consist of the two points $(0, -1)$ and $(0, 3)$. Then, while M-A-B is connected, there is no single arc of M separating A and B in M.

Lemma. *If* J *and* C *are simple closed curves,* A_1 *and* A_2 *are points common to* J *and* C*,* A_1XA_2 *is an arc of* C *such that* $<A_1XA_2>$[8] *has no point in common with* J*,* BY *and* DZ *are arcs such that (1)* $<A_1XA_2>$ *contains*

B and D, (2) *J-A_1-A_2 contains Y and Z,* (3) *the sets $<BY>$ and $<DZ>$ contain no point of the set $J + A_1XA_2$,* (4) *there is a region*[9] *R_1 containing B such that every point of $<BY>$ which lies in R_1 lies in the interior of C,* (5) *there is a region R_2 containing D such that every point of $<DZ>$ which lies in R_2 lies in the exterior of C, then Y and Z separate A_1 and A_2 on the simple closed curve J.*

Proof.—Let R denote the interior or exterior of J according as $<A_1XA_2>$ lies in the interior or exterior of J. Since BY and DZ have only the points Y and Z on J, the domain R contains $BY>$ and $DZ>$. Let R' and R'' be the two domains of $R - <A_1XA_2>$ and let R' be the one whose boundary is the arc A_1YA_2 of J plus the arc A_1XA_2 of C. If Y and Z do not separate A_1 and A_2 on J, the set $<A_1YA_2>$ of J contains the point Z and R' contains $<DZ>$. If p is any point of $<A_1XA_2>$, there exists a region R_p containing the point p but containing no point of the set $J + C - <A_1XA_2>$. If $B = D$, there exists a region R_3 which contains B and is a subset of R_B, R_1 and R_2.[10] There exists an arc B_1WB_2 such that (1) B_1 and B_2 are on $<A_1XA_2>$, (2) $<B_1WB_2>$ is common to R' and R_3, (3) of the two arcs into which B_1 and B_2 divide $J' =$ arc A_1XA_2 of $C +$ arc A_1YA_2 of J, that one which contains B lies in R_3.[11] Then the arc B_1BB_2 of J' is a subset of $<A_1XA_2>$, and B_1 and B_2 separate B and Y and separate D and Z on the simple closed curve J'. Further $<B_1WB_2>$ contains a point P_1 of $<BY>$ and a point P_2 of $<DZ>$.[12] Since $<B_1WB_2>$ is a subset of R' and R_B, $<B_1WB_2>$ contains no point of C. As $<B_1WB_2>$ is contained in R_1 and R_2 the point P_1 belongs to the interior of C and P_2 belongs to the exterior of C. Thus $<B_1WB_2>$ is a connected set containing a point of the interior of C and a point of the exterior of C but containing no point of C. Clearly this is impossible. If $B \neq D$, for any point p of the subarc BD of A_1XA_2 let R'_p be defined as follows: for $p = B$, let R'_p be a region containing B and common to the regions R_1 and R_B;[10] for $p = D$, let R'_p be a region containing D and common to the regions R_2 and R_D;[10] for $B \neq p \neq D$, let R'_p be R_p. For each point p of BD there exists an arc β_p with end-points U_p and V_p on A_1XA_2 in the order $A_1U_pV_pA_2$ such that (1) $<\beta_p>$ is common to R' and R'_p, (2) of the two arcs into which U_p and V_p divide J' that one which contains p lies in R'_p.[11] By the Borel theorem there exists a chain $<U_1V_1>$, $<U_2V_2>$, $<U_3V_3>, \ldots, <U_mV_m>$ of the segments[8] $[<U_ppV_p>]$, such that (1) $<U_1V_1>$ contains B and $<U_mV_m>$ contains D, (2) $<U_iV_i>$ and $<U_jV_j>$ have no points in common unless $i = j \pm 1$. Let k_1 be the largest integer $i (2 \leqq k_1 \leqq m)$, such that $<B_1>$ and $<B_i>$ have a point in common. On the arc β_1 in the order from U_1 to V_1 let T_1 be the first point of $<\beta_{k_1}>$. Let k_2 be the largest integer i such that $<\beta_{k_1}>$ and $<\beta_i>$ have a common point. Let T_2 be the first point of $<\beta_{k_2}>$ on either the subarc $T_1U_{k_1}$ or $T_1V_{k_1}$ of β_{k_1} in the order T_1 to U_{k_1} or T_1 to V_{k_1}. Continue this process.

After a finite number of steps we reach a point T_r which belongs to $<\beta_m>$. Then Λ = subarc U_1T_1 of β_1 + subarc T_1T_2 of $\beta_{k_1}+\ldots+$ subarc $T_{r-1}T_r$ of $\beta_{k_{r-1}}$ + subarc T_rV_m of β_m is an arc having much the same properties as the arc B_1WB_2. We may obtain a contradiction using the arc Λ as we did with the arc B_1WB_2 in the case where $B = D$.

THEOREM 1. *If M is a bounded continuous curve, A and B are distinct points of M and there is just one component of M-A-B that has both A and B as limit points, then M contains an arc which separates A and B in M.*

Proof.—The set $M(A + B)$ is a continuous curve[13] and neither A nor B is a cut-point of $M(A + B)$.[14] Suppose $M(A + B)$ contains a cut-point P. Then $M(A + B) - P$ consists of just two components, one of which contains A and the other contains B.[15] Hence P separates A and B in $M(A + B)$, and thus in M, and is the required arc of the theorem.

Now let us consider the case in which $M(A + B)$ contains no cut-point. In this case the boundary of every complementary domain of $M(A + B)$ is a simple closed curve.[16] Suppose there is a complementary domain D of $M(A + B)$ whose boundary J contains both A and B. Let X and Y be two points of J separating A and B on J. Since there is just one component of M-A-B that has both A and B as limit points, there exists an arc α of M-A-B with end-points X and Y.[17] In the order from X to Y let X' be the last point of the arc AXB of J on α, and let Y' be the first point of the arc AYB of J on the subarc $X'Y$ of α. There is a complementary domain R of M which is a subset of D and such that every boundary point of D is also a boundary point of R.[18] There exists an arc β with end-points X' and Y' and lying in R except for these two points.[19] It is easy to show that β plus the subarc $X'Y'$ of α is a simple closed curve separating A and B in the plane. In this case the subarc $X'Y'$ of α separates A and B in M. For the remainder of the argument we will suppose that there is no complementary domain of $M(A + B)$ whose boundary contains both A and B.

There exists a circle C separating A and B in E_2 and containing at least one point not belonging to M. Let S_1, S_2, S_3, ..., be the set of all components of the common part of C and E_2-M. There is at least one segment in this set and the set is either finite or countable. Each of the sets S_i is a segment belonging to some complementary domain D_i of $M(A + B)$, and let X_i and Y_i be the end-points of S_i. The domains D_i are not necessarily different for different values of i. Let J_i be the boundary of D_i. At least one of the two arcs of J_i from X_i to Y_i contains neither A nor B since the boundary of no complementary domain of $M(A + B)$ contains both A and B. Let α_i be an arc of J_i with end-points X_i and Y_i and containing neither A nor B. Let

$$G = (C - \sum_{i=1}^{\infty} S_i) + \sum_{i=1}^{\infty} \alpha_i$$

Since only a finite number of the curves J_i are of diameter greater than a preassigned positive number,[19] it follows that G is a continuous curve and it is evident from the definition of G that M contains G. If A and B are not separated in E_2 by G, there exists an arc γ with end-points A and B and containing no point of G. Both cases being alike, we will assume that A is in the interior of C. Let order be defined on γ as being from A to B. Let U_1 be the first point of C on γ. The point U_1 belongs to some segment S_{k_1}. Let V_1 be the last point of S_{k_1} on γ. If the subarc V_1B of γ contains any point of C except V_1, let U_2 be the first such point. There is a segment S_{k_2} ($k_2 \neq k_1$) containing U_2. Let V_2 be the last point of S_{k_2} on γ. After a finite number of steps we reach a point V_m such that the subarc AV_m of γ contains every point of C that lies on γ. Let $V_0 = A$ and $U_{m+1} = B$. Let

$$\eta = \sum_{i=0}^{m} \text{subarc } V_iU_{i+1} \text{ of } \gamma + \sum_{i=1}^{m} \text{arc } U_iV_i \text{ of segment } S_{k_i}.$$

The set η is an arc with end-points A and B and having no point in common with G. Now let order be defined on η as being from A to B. It is evident that there is some integer j ($1 \leqq j \leqq m$) such that the segment $<V_{j-1}U_j>$ of η lies in the interior of C and the segment $<V_jU_{j+1}>$ of η lies in the exterior of C. On the subarc BV_j of η let W_2 be the first point of J_{k_j}, and on the subarc AU_j of η let W_1 be the last point of J_{k_j}. The subarcs U_jW_1 and V_jW_2 of η satisfy the conditions of the Lemma. Hence W_1 and W_2 separate X_{k_j} and Y_{k_j} on the curve J_{k_j}. By definition one of the arcs of J_{k_j} from X_{k_j} to Y_{k_j} belongs to G. This arc α_{k_j} contains one of the points W_1 or W_2. Then η *does* contain a point of G. Therefore, G separates A and B in the plane E_2. Let D be the complementary domain of G containing A. Let L be the outer boundary of D relative to the point B.[20] Then L is a simple closed curve which is a subset of G, and thus of M, and separates A and B in the plane E_2.

Since only a finite number of the curves J_i contain a point whose distance from C is greater than a given positive number,[19] the set L contains an arc A_1ZA_2 which lies on the boundary of some complementary domain D_s of $M(A + B)$. The domain D_s contains a complementary domain D_s' of M whose boundary contains the arc A_1ZA_2.[18] There is an arc A_1WA_2 whose end-points are A_1 and A_2 and which lies in D_s' except for A_1 and A_2.[19] The two cases being alike, we will assume that $<A_1WA_2>$ lies in the complementary domain of L that contains B. Let L_1 be the boundary of the complementary domain of $L + A_1WA_2$ that contains B. The set L_1 is a simple closed curve separating A and B in the plane and having one of the two arcs of L from A_1 to A_2 in common with M. This is the desired arc of M separating A and B in M.

THEOREM 2. *If A and B are distinct points of a bounded continuous* *curve M, there exists a simple closed curve separating A and B in the plane and having in common with M a set of n arcs, where n is the number of components of M-A-B that have both A and B as limit points.*

Proof.—I. If $n = 1$, then by theorem 1 there is either a point or an arc that separates A and B in M. If A and B are separated in M by a point P, there exists a simple closed curve separating A and B in E_2 and having only the point P in common with M.[21] In the other case we showed in the proof of theorem 1 the existence of a simple closed curve separating A and B in E_2 and having just one arc in common with M.

II. If $n > 1$, let $H_1, H_2, \ldots, H_n$ denote the set of components of M-A-B that have both A and B as limit points. By theorem 1 each component H_i contains an arc α_i that separates A and B in $H_i + A + B$. There exists a set of n complementary domains of M, $D_1, D_2, \ldots, D_n$ such that (1) both A and B are boundary points of each D_i, (2) each domain D_i has points of just two components of the set $[C_j]$ on its boundary, (3) each component C_j contains boundary points of just two domains of the set $[D_i]$. We may suppose that for each i, C_i and C_{i+1} contain boundary points of D_i.[22] It is easy to see that α_i contains a boundary point q_i of D_i and α_{i+1} contains a boundary point p_{i+1} of D_i. Let β_i be an arc whose end-points are q_i and p_{i+1} and such that D_i contains $<\beta_i>$. Then

$$\sum_{i=1}^{n} (\beta_i + \text{subarc } p_i q_i \text{ of } \alpha_i)$$

is a simple closed curve separating A and B in E_2 and having exactly n arcs in common with M.

[1] Presented to the American Mathematical Society, February 26, 1927. As presented to the Society, this paper contained several results which are not included in it in its present form. I found that one of these results had been previously stated by C. M. Cleveland. See these PROCEEDINGS, **13**, 1927 (275–276). The other results will be published as a part of another paper, "Concerning the Arc-Curves and Basic Sets of a Continuous Curve," which will appear in the *Trans. Amer. Math. Soc.*

[2] *Fund. Math.*, **7**, 1925 (302–307).

[3] This condition can be replaced by the condition that M is not the entire plane and the boundary of every complementary domain of M is bounded.

[4] In this paper a single point is considered as a special case of an arc.

[5] A connected subset H of a point set K is said to be a *component* of K if there is no connected subset of K containing H as a proper subset.

[6] See my paper referred to in footnote 1. Hereafter I shall refer to this paper as "Arc-Curves."

[7] However, it is likely that the theorem will hold under certain restrictions if we assume in addition that A and B are connected.

[8] If XYZ denotes an arc with end-points X and Z, the symbols $<XYZ$, $XYZ>$ and $<XYZ>$ denote XYZ-X, XYZ-Z and XYZ-X-Z, respectively. By the *segment* XYZ is meant XYZ-X-Z.

[9] We may interpret *region* as the interior of a simple closed curve. See however,

Moore, R. L., *Trans. Amer. Math. Soc.*, **17**, 1916 (131–164). Hereafter we will refer to this paper as "Foundations."

[10] "Foundations," theorem 6.

[11] *Ibid.*, theorem 28.

[12] *Ibid.*, theorem 29.

[13] If K is a closed subset of M, $M(K)$ is a continuous curve. See "Arc-Curves," theorem 7.

[14] "Arc-Curves," theorem 8.

[15] If P is a cut-point of $M(K)$, every component of $M(K) \cdot P$ contains at least one point of K. See "Arc-Curves," theorem 9, part (1).

[16] Whyburn, G. T., these PROCEEDINGS, **13**, 1927 (31–38), theorem 10.

[17] Moore, R. L., *Math. Zeit.*, **15**, 1922 (255).

[18] "Arc-Curves," theorem 10, part (5).

[19] Schoenflies, A., *Die Entwicklung der Lehre von den Punktmannigfaltigkeiten*, Zweite Teil, Leipzig, **1908** (237).

[20] Moore, R. L., these PROCEEDINGS, **11**, 1925 (469–476), footnote 5.

[21] See an abstract of a paper by R. G. Lubben, "The Separation of Mutually Separated Subsets of a Continuum by Curves," *Bull. Amer. Math. Soc.*, **32**, 1926 (114).

[22] Subscripts are reduced modulo n.

ON SOME PROPERTIES OF ONE-VALUED TRANSFORMATIONS OF MANIFOLDS

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1. *Some Constants of a Transformation.*—Given any set of ν-cycles in an n-complex μ^n, its maximum number of independent cycles with respect to homologies shall be called the "rank" of the set; it is not greater than the νth Betti number π_ν of μ^n. Consider another n-complex M^n transformed into μ^n by a one-valued continuous transformation f and the set of all $f(c_i^\nu)$, where the c_i^ν are the ν-cycles of M^n; then the rank r_ν of this set is a constant of f ($\nu = 0, 1, \ldots, n$).

If $c_1^\nu, c_2^\nu, \ldots, c_{p_\nu}^\nu$, where p_ν is the νth Betti number of M^n, form a fundamental set in M^n and $\gamma_1^\nu, \gamma_2^\nu, \ldots, \gamma_{\pi_\nu}^\nu$ a fundamental set in μ^n, then the transformations of the ν-cycles define a system of homologies[1]

$$f(c_j^\nu) \approx \sum_{k=1}^{\pi_\nu} \alpha_{jk}^\nu \gamma_k^\nu \quad (j = 1, 2, \ldots, p_\nu) \tag{1}$$

and r_ν is the rank of the matrix

$$A_\nu = \| \alpha_{jk}^\nu \|.$$

When the c_j^ν and γ_k^ν are replaced by other fundamental sets, A_ν is trans-