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A_n theory, L.S. category, and strong category

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Abstract Relations between category and strong category are studied. The notion of a homotopy coalgebra of order r over the Ganea comonad is introduced. It is shown that cat(X) =Cat(X) holds if a finite 1-connected complex X carries such a structure with r sufficiently large.

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1 Introduction and statement of results

The Lusternik-Schnirelmann category cat(X) of a topological space X is defined to be the least number n such that there are $n+1$ open subsets which cover X and which are contractible in X. A classical theorem of Lusternik and Schnirelmann states that cat(M)$+1$ is a lower bound for the number of critical points of a smooth function f on the smooth manifold M [34]. Due to this result, the invariant cat plays an important role in geometry and analysis [19], [31]. The strong category Cat(X) was introduced by Fox [21] as an approximation to cat(X). It is the smallest number n such that there is a space of the homotopy type of X which can be covered by $n+1$ open contractible subsets. There are variants of these notions which give the same values on pointed path connected CW-complexes [2], [57]. For cat, a first one, given by Whitehead, defines cat(X) $\leq n$ if the diagonal

$$\Delta^{n+1} : X \longrightarrow X^{n+1}$$

factors up to homotopy through the fat wedge $X^{\langle n+1 \rangle}$. A second one, due to Ganea, puts cat(X) $\leq n$ if the homotopy fibration

$$\Omega X^{*n+1} \rightarrow B_n\Omega X \rightarrow B_\infty \Omega X \simeq X$$

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admits a homotopy section. Here ΩX^{*n+1} is the $(n+1)$-fold join of the Moore loop space ΩX and $B_n\Omega X$ stands for the nth stage of the classifying space of the monoid ΩX. The strong category can be described as cone-length. That is, $\mathrm{Cat}(X) \leq n$ if and only if there are cofibration sequences

$$L_i \to X_i \to X_{i+1}, \quad 0 \leq i < n$$

with $X_0 \simeq *$ and $X_n \simeq X$ [24]. Later it was proved by Cornea [11], [12], that one can find always such an n-cone presentation in which $L_i \simeq \Sigma^i Z_i$.

In this paper, we will study relations between $\mathrm{cat}(X)$ and $\mathrm{Cat}(X)$ of a topological space X. In general there is an inequality [24]:

$$\mathrm{cat}(X) \leq \mathrm{Cat}(X) \leq \mathrm{cat}(X) + 1.$$

Until recently there were no spaces X known with $2 \leq \mathrm{cat}(X) \neq \mathrm{Cat}(X)$. This has now changed. The first example, found by Dupont [18], is a rational space with $\mathrm{cat}(X) = 3$ and $\mathrm{Cat}(X) = 4$. There can be no rational example of lower category [20]. A little later Stanley constructed, for each n, spaces X_n with $\mathrm{cat}(X_n) = n$ and $\mathrm{Cat}(X_n) = n+1$ [49]. Stanley's work was inspired by the ideas used by Iwase in the construction of his counterexample to the Ganea conjecture [28]. In a certain range both invariants agree. For example, Clapp and Puppe, improving on an earlier result of Ganea [24], showed [9] that, if X is a finite $(l-1)$-connected $(2 \leq l)$ CW-complex with $(X) \leq (2\mathrm{cat}(X)+1)l-3$, the equality $\mathrm{cat}(X) = \mathrm{Cat}(X)$ holds. The case of category one was studied by Ganea in [22]. The condition $\mathrm{cat}(X) = 1$ is equivalent to X being a co-H space and $\mathrm{Cat}(X) = 1$ translates into X being of the homotopy type of a suspension. For a co-H space X which admits the structure of a coalgebra up to homotopy over the cotriple defined by the functor $\Sigma\Omega$ a theorem in [22] tells us that already the inequality $\dim(X) \leq 4l-5$ implies that $\mathrm{Cat}(X) = 1$. Later Saito showed that this bound can be further improved for co-A_4 spaces [42]. These results were generalized independently by Arkowitz and Golasinski [1] on the one hand and by Klein, Schwänzl and Vogt on the other [32]. Both groups apply a version of A_∞ theory for co-H spaces. In contrast to [32] where the comultiplication is the main object of study, Arkowitz and Golasinki choose to work with a section for the first Ganea fibration instead. Following a suggestion of Ganea in [22] the notion of a homotopy coalgebra of order r over $\Sigma\Omega$ is introduced which plays the part of an A_r space structure. The main theorem in [1] is then that a finite 2-connected co-H space whose co-H structure satisfies all higher coherencies can be desuspended. In [32] the same result is achieved. Apart from the connectivity hypothesis, this establishes a perfect Eckmann-Hilton dual result to Stasheff's classical delooping theorem for A_∞ spaces.

It is the aim of this article to generalize some of these results for $\mathrm{cat}(X) \geq 2$. In doing so we will follow the route laid out by Arkowitz and Golasinski insofar as we study sections of the nth Ganea fibration instead of compressions of the diagonal into the fat wedge. The functor $\Sigma\Omega$ is generalized by the nth Ganea space functor $G_n(-)$ defined by Ganea's fibre-cofibre construction [24] (or, equivalently up to homotopy, $B_n\Omega$). This functor defines a cotriple as was shown by Deligiannis [13]. The key notion will be that of a (weak) homotopy coalgebra of order r for this cotriple. Homotopy coalgebras of order 1 over the Ganea cotriple were studied by Deligiannis in his thesis [14]. These make up the second layer of structure. On the other hand, we follow the authors of [32] in that we use higher doses of A_n theory in a form which was pioneered by Boardman and Vogt [6].

The following theorem is an exact generalization of Ganea's theorem quoted above.

Theorem 1.1 *Let X be of the homotopy type of an $(l-1)$-connected $(l \geq 2)$CW-complex with $cat(X) = n$ and $dim(X) \leq 2(n+1)l - 5$. Then $Cat(X) = n$ if and only if X is a homotopy coalgebra of order 1 over $B_n\Omega$.*

The next theorem generalizes the main results in [1] and [9] from $n = 1$ to all n and from $r = 1$ to all r respectively .

Theorem 1.2 *Let X be of the homotopy type of a CW-complex. Suppose that cat $(X) = n$ and that X admits a weak homotopy coalgebra structure over $B_n\Omega$ of order r. Furthermore, suppose that X is $(l-1)$-connected $(l \geq 2)$ and*

$$dim(X) \leq (2n+1)l + (r-1)(n(l-1)-1) - 3.$$

Then $cat(X) = Cat(X)$.

As was shown by Cornea in [11], there exists for every space X with $\mathrm{cat}(X) = n$ a space Z such that $\mathrm{Cat}(X \vee \Sigma^n Z) \leq n$. So in a sense we are looking for conditions which make it possible to cancel $\Sigma^n Z$.

The guiding idea of this paper and for the proof of these theorems may be summarized as follows. Let Y be a space with $\mathrm{cat}(Y) \leq n$. The first place to start looking for an n-cone approximation is a section $\sigma : Y \to B_n\Omega Y$. Under certain conditions on the connectivity and dimension of Y this approximation can be used to construct an n-cone decomposition for Y. Now, by a mild variation of an argument of Clapp and Puppe which appears in Sect. 4. as Lemma 4.3, the space ΩY can be replaced by any A_n space D. So in case the connectivity of σ is not high enough new candidates of A_n spaces must be found. These searched for spaces show up as classifying spaces $B_nD_i(X)$ of A_n spaces $D_i(X)$ if σ can be lifted to a homotopy coalgebra structure of order i. It will be shown that the connectivity of the structure maps $\gamma_i : X \to B_nD_i(X)$ grow with i. Since the spaces $B_nD_i(X)$ are all n-cones, a homotopy coalgebra of order i gives rise to n-cone approximations of X which improve with i. To make this work, we have first of all to construct the A_n structure on $D_i(X)$, which is defined as a homotopy pullback along A_n maps. Moreover, we must investigate the homotopy type of the spaces $B_nD_i(X)$. A complication arises from the fact that a certain natural map η which figures in the definition of a homotopy coalgebra is only an A_n map, not an A_n homomorphism. This means that η does not commute with the A_n actions on the nose but only up to coherent homotopy. To settle this, the approach of Boardman and Vogt to A_n theory is best suited for the following reason. In A_∞ theory one studies monoids up to homotopy equivalence. The correlation between the homotopy multiplicative and a related strict multiplicative structure is the key to the problem just mentioned in the A_n case. Here the strict structure is that of a partial monoid. Now Boardman and Vogt join two homotopy operations through deformation to a strict operation. This is expressed through their cubical subdivision of the associhedra and that structure makes the comparison between homotopy and strict structure more easy. Connections between cat and A_∞ theory already played a role in Iwase's work on Ganea's conjecture [28], [29].

This article is organized as follows. In Sect. 2 we define homotopy coalgebras over the Ganea comonad and look at examples of spaces with and without such a structure. In Sect. 3 we study how the construction B_n behaves with respect to homotopy pullbacks. Given the work in the appendix, the results are more or less straightforward generalizations of lemmas due to Ganea in the case $n = 1$. The connectivity estimates for the maps γ_i are deduced in this section. With the help of Lemma 4.3 these are used to prove our main results Theorems 1.1. and 1.2. in Sect. 4. Finally, we quote some open problems. In order to make the paper more readable, we have packaged the needed results on A_n theory in a long appendix which contains three sections. In Sect. A.1 we recall some basics on A_n theory in the way of Boardman and Vogt. The main result of Sect. A.2 is Theorem 2.1 which says that the homotopy pullback of an A_n map along a monoid map carries an induced A_n structure. This theorem is used in the construction of the spaces $B_n D_i(X)$. The proof uses the notion of an A_n rectification M_n of an A_n space X. The space M_n is a partial monoid homotopy equivalent to X. Section A.3 is devoted to classifying spaces of partial monoids. In case the partial monoid is the rectification M_n of an A_n space, we determine the homotopy type of the nth stage of the classifying space. The arguments and results here are all adaptations of Stasheff's classical work in [50]. These adaptions are needed since for the proof of Theorem 2.1 we found it necessary to work with partial monoids and to use the tree approach of Boardman and Vogt to A_n theory. The multiplication on a partial monoid is strictly associative but not everywhere defined and hence is not an A_n-form. Of course, both structures are closely related as is shown in Sect. A.2.

2 Homotopy coalgebras over $B_n\Omega$

We will work in the ground category *Top* of compactly generated spaces in the sense of [56]. Let Top_* denote the category of based compactly generated spaces and maps. The categories *Top* and Top_* are closed monoidal, complete and cocomplete.

For the used notions and results on A_n theory the appendix can be consulted. The following theorem is essential for the definition of a homotopy coalgebra over $B_n\Omega$ to make sense. For $n \leq 3$ it appears in [59, Sect. 2.5.3.], and a more general theorem is stated without proof in [30], but we could not find a proof for the general case in the literature. The proof is given in the Appendix.

Theorem 2.1 *Let (X, F) be a well pointed A_n space and $g : X \to N$ an A_n map to a monoid N. Moreover, let $f : K \to N$ be a map of monoids. Then in the homotopy pullback square*

$$\begin{array}{ccc} P & \xrightarrow{\tilde{g}} & K \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{g} & N. \end{array}$$

there exists an induced A_n structure on the space P such that the maps $\tilde{f}$ and $\tilde{g}$ induce maps of partial monoids between the rectifictions. Consequently, there are also induced maps on classifying spaces.

Let D be an A_n space. The map η_D given by the composition

$$D \to \Omega\Sigma D \to \Omega B_n D$$

is an A_n map [51, Sect. 11.10]. It plays a prominet role in the following definition, which is central for this paper.

Definition 2.2 A *weak homotopy coalgebra* struture over $B_n\Omega$ of order $r \leq \infty$ on X is given by the following data:

(a) A sequence of A_n spaces $D_0(X), \ldots, D_r(X)$ where $D_0(X) = *$.
(b) Maps $\gamma_i : X \to B_nD_i(X) \quad i \leq r$.

Such that:

i) The space $D_i(X)$ is given inductively as the following homotopy pullback:

$$\begin{array}{ccc} D_i(X) & \xrightarrow{\tilde{\eta}_{D_{i-1}(X)}} & \Omega X \\ {\scriptstyle \widetilde{\Omega\gamma}_{i-1}}\downarrow & & \downarrow{\scriptstyle \Omega\gamma_{i-1}} \\ D_{i-1}(X) & \xrightarrow{\eta_{D_{i-1}(X)}} & \Omega B_nD_{i-1}(X). \end{array}$$

Here the A_n struture on $D_i(X)$ is the one corresponding to the induced A_n rectification as explained in the Appendix.

ii) The maps γ_i, for $i \geq 1$, are to be homotopy sections of the following composition:

$$B_nD_i(X) \overset{B_n\tilde{\eta}_{D_{i-1}(X)}}{\longrightarrow} B_n\Omega(X) \overset{\mathcal{E}_{\mathcal{X}}}{\longrightarrow} X.$$

Here $\mathcal{E}_{\mathcal{X}}$ is the composition of $B_n\Omega X \to B_\infty\Omega X$ with some fixed homotopy equivalence $B_\infty\Omega X \simeq X$.

If in addition the following diagram commutes up to homotopy for $i \leq r$ we will call such a structure a *genuine homotopy coalgebra* over $B_n\Omega$ of order r:

$$\begin{array}{ccc} X & \xrightarrow{\gamma_i} & B_nD_i(X) \\ {\scriptstyle \gamma_1}\downarrow & & \downarrow{\scriptstyle B_n\eta_{D_i(X)}} \\ B_n\Omega X & \xrightarrow{B_n\Omega\gamma_i} & B_n\Omega B_nD_i(X). \end{array}$$

We quote some results from the literature. The first two which follow are special cases of theorems proved in [1, Sects.1.5.,1.6.].

Proposition 2.3 *A weak homotopy coalgebra of order r is a genuine homotopy coalgebra of order $r-1$.*

Proposition 2.4 *Suppose X is of the homotopy type of a space of the form B_nA with A_n space A. Then X carries a genuine homotopy coalgebra structure of infinite order.*

Recall that the Ganea space $G_n(X)$ of X is defined as follows. First put $G_0(X) = *$. If $p_{n-1} : G_{n-1}(X) \to X$ is defined one puts $G_n(X) = G_{n-1}(X) \cup C(F_{n-1})$ and extends to $C(F_{n-1})$ by $p_n(b, w, t) = w(0)$. Here F_{n-1} is the homotopy fibre of p_{n-1}. Note that in contrast to Ganea we do not turn p_n into a fibration. We remind the reader that $G_n(X)$ and $B_n\Omega X$ are homotopy equivalent in a natural way under mild assumptions on X.

The next theorem is due to Deligiannis and appears in his thesis [14, Sect. 5.2.2]. Deligiannis shows in [13] that the functor G_n defines a comonad. Hence it makes sense to talk about coalgebras over this comonad.

Theorem 2.5 *Let X be a path connected CW-complex with $Cat(X) = n$. Then X is of the homotopy type of a strict coalgebra over the comonad G_n.*

Corollary 2.6 *Let X be as above. Then X carries a genuine homotopy coalgebra structure of order* 1 *over $B_n\Omega$.*

Not every space of category n carries the structure of a weak coalgebra of order > 1.

Example 1 In [18] a 3-connected rational space X of dimension 29 is constructed by Dupont with $\mathrm{cat}(X) = 3$ and $\mathrm{Cat}(X) = 4$. It follows from Theorem 1.2 that there is no weak homotopy coalgebra structure over $B_3\Omega$ of order 2 on X.

Example 2 In this example we work with p-local spaces where p is an odd prime. Let $\omega_n \in \pi_{2np-3}(S^{2n-1})$ be the first nontrivial element in the kernel of the double suspension $n > 1$. Consider

$$Q_n = (S^2)^{<n>} \cup e^{2np-2}$$

where $(S^2)^{<n>}$ is the nth fat wedge of S^2 and the $(2np-2)$-cell is attached by the composition of the universal Whitehead product of order n w_n and ω_n. For dimensional reasons the pth James-Hopf invariant H_p of ω_n vanishes. It follows from [25, Theorem 1] that ω_n is a co-H map. So the the main result of [49] applies and gives us $\mathrm{cat}(Q_n \times S^2) = \mathrm{cat}(Q_n) = n$ and $\mathrm{Cat}(Q \times S^2) = n+1$. Note that $Q \times S^2$ is a 1-connected space of dimension $4np-4$. Apply Theorem 1.2 to see that $Q_n \times S^2$ admits no weak homotopy coalgebra structure over $B_n\Omega$ of order r if $r(n-1) \geq 4np-3n-4$.

3 B_n and homotopy pullbacks

For $n = 1$, the results of this section appear in Ganea's classical paper [22]. The connectivity of a map $f : X \to Y$ between connected spaces will be written $\mathrm{Con}(f)$, it is defined as the largest integer n such that

$$f_* : \pi_i(X) \to \pi_i(Y)$$

is an isomorphism for $i < n$ and an epimorphism for $i = n$ [58]. For a space X, one puts $Con(X) = Con(X \to *)$.

Lemma 3.1 *Let $f : X \to Y$ be a map with $Con(f) = m$, $Con(X) \geq l-1 \geq 1$ and $m \geq l-1$. Furthermore, let σ be a homotopy section of $p_n : G_n(Y) \to Y$. Consider*

$$\begin{array}{ccccc} P & \xrightarrow{\theta} & G_n(X) & \xrightarrow{p_n} & X \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle G_n(f)} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{\sigma} & G_n(Y) & \xrightarrow{p_n} & Y \end{array}$$

where P is the homotopy pullback along σ.

Under these assumptions, we have:

$$Con(p_n\theta) \geq m + n(l-1).$$

Proof Let $i : F_{G_nf} \to F_f$ denote the map induced on the homotopy fibres. Since σ is a homotopy section of p_n, it follows from the five lemma that $\mathrm{Con}(p_n\theta) = \mathrm{Con}(i)$.

Recall that a Neisendorfer diagram [38] is a three by three diagram like the one below in which the rows and columns are fibration sequences, and the the maps between fibers are induced by the maps in the bottom right square. Consideration of the following Neisendorfer diagram based on the right lower square

$$\begin{array}{ccccc}
F_{F_n(f)} & \longrightarrow & F_{G_n(f)} & \xrightarrow{i} & F_f \\
\downarrow & & \downarrow & & \downarrow \\
F_n(X) & \longrightarrow & G_n(X) & \longrightarrow & X \\
\downarrow{\scriptstyle F_n(f)} & & \downarrow{\scriptstyle G_n(f)} & & \downarrow{\scriptstyle f} \\
F_n(Y) & \longrightarrow & G_n(Y) & \longrightarrow & Y
\end{array}$$

shows that Con(i) =Con($F_n(f)$). Since the homotopy equivalence $F_n(X) \simeq \Sigma^n(\Omega(X)^{\wedge n+1})$ [23] is natural up to homotopy we find

$$\mathrm{Con}(F_n(f)) = \mathrm{Con}(\Sigma^n(\Omega(f)^{\wedge n+1})) = m + n(l-1).$$

□

In preparation for the proof of the next result, recall [10] that a map of n-cone decompositions (or level-preserving map) $f : X \to X'$, is defined by maps and commuting diagrams $0 \le i \le n-1$

$$\begin{array}{ccccc}
Z_i & \longrightarrow & X_i & \longrightarrow & X_{i+1} \\
\downarrow{\scriptstyle g_i} & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i+1}} \\
Z'_i & \longrightarrow & X'_i & \longrightarrow & X'_{i+1},
\end{array}$$

where the rows are cofibration sequences, f_{i+1} is an induced map on cofibres, $X_0 = X'_0 = *, X_n \simeq X, X'_n \simeq X'$, and f_n and f are homotopy equivalent in the sense that there is a commutative square joining them by homotopy equivalences. The spaces X_i, Z_i and the maps f_i, g_i are called the components of X and f respectively. In the proposition below all spaces are assumed to be of the homotopy type of a CW-complex.

Proposition 3.2 *Let $\alpha : A \to C$ be an A_n map and $\beta : B \to C$ a monoid map, and consider the homotopy pullback square*

$$\begin{array}{ccc}
D & \xrightarrow{\tilde{\alpha}} & B \\
\downarrow{\scriptstyle \tilde{\beta}} & & \downarrow{\scriptstyle \beta} \\
A & \xrightarrow{\alpha} & C
\end{array}$$

Suppose that

(a) $Min(Con(A), Con(B)) := l - 1 \geq 1$
(b) $Min(Con(\alpha) = p, Con(\beta) = q) \geq l - 1$

Then the map ϕ_n into the homotopy pullback V which is induced by $B_n\tilde{\alpha}$ and $B_n\tilde{\beta}$

$$\begin{array}{ccccc} B_nD & \xrightarrow{\phi_n} & V & \xrightarrow{\widetilde{B_n\alpha}} & B_nB \\ & & \downarrow{\scriptstyle \widetilde{B_n\beta}} & & \downarrow{\scriptstyle B_n\beta} \\ & & B_nA & \xrightarrow{B_n\alpha} & B_nC \end{array}$$

has $Con(\phi_n) \geq p + q + 1$.

Proof Consider the commutative diagram:

$$\begin{array}{ccccc} B_nD & \xrightarrow{B_n\tilde{\alpha}} & B_nB & \longrightarrow & B_nB/B_nD \\ \downarrow{\scriptstyle \phi_n} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \varphi_n} \\ V & \xrightarrow{\widetilde{B_n\alpha}} & B_nB & \longrightarrow & B_nB/V \\ \downarrow{\scriptstyle \widetilde{B_n\beta}} & & \downarrow{\scriptstyle B_n\beta} & & \downarrow{\scriptstyle \psi_n} \\ B_nA & \xrightarrow{B_n\alpha} & B_nC & \longrightarrow & B_nC/B_nA. \end{array}$$

We will use a special case of Serre's relative theorem [9, A.1] which states that if $\text{Con}(\alpha) = p$ and $\text{Con}(\beta) = q$ then $\text{Con}(\theta) \geq p + q + 1$ for the map

$$\theta : B/D \to C/A$$

induced by $\beta, \tilde{\beta}$. This gives us:

$$\text{Con}(\psi_n) \geq \text{Con}(B_n\alpha) + \text{Con}(B_n\beta) + 1 \geq p + q + 3.$$

Since $B_n\tilde{\alpha}$ and $B_n\beta$ are maps of n-cones, Lemma 3.1 in [10] tells us that the cofibres B_nB/B_nD and B_nC/B_nA are also n-cones such that the collapsing map is level preserving. The map $\gamma_n := \psi_n\varphi_n$ is induced by the maps $B_n\beta$ and $B_n\tilde{\beta}$. From this and Proposition A.3.4, it follows that γ_n is a map of n-cones with components

$$\gamma_i : B_iB/B_iD \to B_iC/B_iA$$
$$\bar{\gamma}_i : B^{*i+1}/D^{*i+1} \to C^{*i+1}/A^{*i+1}, i \leq n - 1.$$

We claim that

$$\text{Con}(\gamma_n) \geq p + q + 2.$$

If the claim is true we are done, since then $\text{Con}(\varphi_n) \geq p + q + 2$ holds by the estimate for $\text{Con}(\psi_n)$ above and $\text{Con}(\phi_n) \geq p + q + 1$ follows with the Five lemma. The start

$n = 1$ is in [22, Proof of Lemma 3.1]. An application of the Five lemma shows that it suffices to demonstrate that

$$\mathrm{Con}(\bar{\gamma}_{m-1}) \geq p + q + 1$$

for $m \leq n$. The inequality $\mathrm{Con}(\theta) \geq p+q+1$ implies $\mathrm{Con}(\theta^{*m}) \geq p+q+1+(m-1)(l-1)$ by the Künneth formula and the homotopy equivalence $X^{*i+1} \simeq \Sigma^i X^{\wedge i+1}$.

Consider

$$\begin{array}{ccccc}
B^{*i+1}/D^{*i+1} & \xrightarrow{\;j\;} & (B/D)^{*i+1} & \xrightarrow{\;i\;} & C_j \\
\Big\downarrow \bar{\gamma}_i & & \Big\downarrow \theta^{*i+1} & & \Big\downarrow \hat{\gamma}_i \\
C^{*i+1}/A^{*i+1} & \xrightarrow{\;\bar{j}\;} & (C/A)^{*i+1} & \xrightarrow{\;\bar{i}\;} & C_{\bar{j}}
\end{array}$$

where the lines are cofibration sequences, $j, \bar{j}$ are the natural maps, and $\hat{\gamma}_i$ is the map induced on homotopy cofibres. From this diagram and the Five lemma it follows that it suffices to show that $\mathrm{Con}(\hat{\gamma}_i) \geq p + q + 2$ and we prove this by induction. Let us now identify the homotopy type of C_j. The map j is up to homotopy the inclusion of

$$\Sigma^i(B^{\wedge i+1} \cup C(D^{\wedge i+1}))$$

into

$$\Sigma^i(B^{\wedge i+1} \cup C(D \wedge B \wedge \cdots \wedge B \cup \cdots \cup B \wedge \cdots \wedge B \wedge D)).$$

Hence, there is a homotopy equivalence

$$C_j \simeq \Sigma^{i+1}((D \wedge B \wedge \cdots \wedge B) \cup \cdots \cup (B \wedge \cdots \wedge B \wedge D)/(D^{\wedge i+1})).$$

We do an induction on i. In case $i = 1$ we obtain from the formula above that

$$C_j \simeq \Sigma^2((D \wedge (B/D) \vee (B/D) \wedge D)).$$

Here we have used $(X \wedge Y)/(A \wedge Y) \simeq (X/A) \wedge Y$. Hence, by naturality and symmetry, it suffices to show that $\mathrm{Con}(\Sigma^2(\tilde{\beta} \wedge \theta)) \geq p + q + 2$. This can be seen as follows. Since $\tilde{\beta} \wedge \theta = (1_A \wedge \theta)(\tilde{\beta} \wedge 1_{B/D})$, and because the operation of smash product with a fixed space preserves cofibre sequences, we find that $\mathrm{Con}(\tilde{\beta} \wedge \theta)$ equals

$$\mathrm{Min}(\mathrm{Con}(\tilde{\beta} \wedge 1_{B/D}), \mathrm{Con}(1_A \wedge \theta)) = \mathrm{Min}(\mathrm{Con}(C_{\tilde{\beta}} \wedge 1_{B/D}), \mathrm{Con}(1_A \wedge C_\theta)).$$

But $\mathrm{Con}(C_{\tilde{\beta}} \wedge 1_{B/D}) = \mathrm{Con}(C_\beta \wedge 1_{B/D}) = q + \mathrm{Con}(\tilde{\alpha}) + 1 = p + q + 1$. On the other hand $\mathrm{Con}(1_A \wedge C_\theta) \geq l + p + q + 1$. Hence, $\mathrm{Con}(\tilde{\beta} \wedge \theta) = p + q + 1$ which is more than we need.

For the inductive step, consider any $i > 1$. As above, there is a homotopy equivalence

$$\begin{aligned}
&(D \wedge B \wedge \cdots \wedge B \cup \cdots \cup B \wedge \cdots \wedge B \wedge D)/(D^{\wedge i+1}) \\
&\simeq D \wedge (B^{\wedge i}/D^{\wedge i}) \bigcup_{D^{\wedge 2} \wedge (B^{\wedge i-1}/D^{\wedge i-1})} \cdots \bigcup_{(B^{\wedge i-1}/D^{\wedge i-1}) \wedge D^{\wedge 2}} (B^{\wedge i}/D^{\wedge i}) \wedge D.
\end{aligned}$$

The assertion follows by the inductive hypothesis and an application of the Mayer-Vietoris sequence. □

4 Proof of the main theorems

We will need some more connectivity estimates. The following lemma is well known but we could not find a reference.

Lemma 4.1 *Let D be a $(l-2)$-connected (with $l \geq 2$) A_n space of the homotopy type of a CW-complex. Then the estimate*

$$Con(\eta_D) \geq (n+1)l - 3$$

holds true.

Proof Recall from Proposition A.3.2 that there is a homotopy fibration

$$D \xrightarrow{i} E_{n-1}D \xrightarrow{p} B_{n-1}D$$

with $E_{n-1}D \simeq D^{*n}$ and $B_nD \simeq B_{n-1}D/E_{n-1}D$. Then η_D is up to homotopy the map induced on the homotopy fibres of the vertical maps in the following homotopy pushout diagram:

$$\begin{array}{ccc} E_{n-1}D & \longrightarrow & * \\ \downarrow & & \downarrow \\ B_{n-1}D & \longrightarrow & B_nD. \end{array}$$

Because $\mathrm{Con}(D^{*n}) = nl - 2$ the assertion follows from the Blakers-Massey theorem in the form presented in [8, Sect. 9.7]. □

For a weak coalgebra X of order r let

$$\begin{array}{ccccc} B_nD_{r+1}X & \xrightarrow{\phi_r} & V_r & \xrightarrow{\widetilde{B_n}\eta_{D_rX}} & B_n\Omega X \\ & & \downarrow{\scriptstyle \widetilde{B_n}\Omega\gamma_r} & & \downarrow{\scriptstyle B_n\Omega\gamma_r} \\ & & B_nD_rX & \xrightarrow{B_n\eta_{D_rX}} & B_n\Omega B_nD_rX \end{array}$$

be the diagram of 3.2. Note that D_{r+1} exists as the appropriate homotopy pullback which is an A_n space by Theorem 2.1.

Proposition 4.2 *Let X be a weak coalgebra of degree r. Suppose $Con(X) = l - 1$ and $l \geq 2$. Then the following holds for $s \leq r$:*

1) $\mathrm{Con}(\gamma_s) \geq (n+1)l + (s-1)(n(l-1)-1) - 2$
2) $\mathrm{Con}(D_sX) = l - 2$
3) $\mathrm{Con}(\varepsilon_X \circ \widetilde{B_n}\eta_{D_sX}) = \mathrm{Con}(\gamma_s) + n(l-1)$
4) $\mathrm{Con}(\phi_s) \geq 2(n+1)l + (s-1)(n(l-1)-1) - 5.$

Proof We do an induction on r. Let $r = 1$:

1) The assertion follows from $\varepsilon_X\gamma_1 \simeq id$ since the homotopy fibre of ε_X is $(\Omega X)^{*n+1}$.
2) This is clear because $D_1X = \Omega X$.

3) This point is a consequence of Lemma 3.1. applied to $\sigma = B_n\eta_{\Omega X}$ and $f = \gamma_1$.
4) The assertion follows from Lemma 4.1 (1) and Proposition 3.2 applied to $\alpha = \eta_{\Omega X}$ and $\beta = \Omega\gamma_1$.

Suppose the assertion is true for $r \leq q$:
1) Using [1, 1.6], we have

$$\widetilde{B_n}\eta_{D_qX} \circ \phi_q \circ \gamma_{q+1} \simeq \gamma_q$$

and hence

$$\varepsilon_X \circ \widetilde{B_n}\eta_{D_qX} \circ \phi_q \circ \gamma_{q+1} \simeq id.$$

This gives us

$$\mathrm{Con}(\gamma_{q+1}) = \mathrm{Con}(\varepsilon_X \circ \widetilde{B_n}\eta_{D_qX} \circ \phi_q) - 1 = \mathrm{Min}(\mathrm{Con}(\varepsilon_X \circ \widetilde{B_n}\eta_{D_qX}, \mathrm{Con}(\phi_q)) - 1.$$

Now use the induction hypothesis on ϕ_q and $\varepsilon_X \circ \widetilde{B_n}\eta_{D_qX}$ and note that the connectivity of the former is greater or equal to that of the later since $l \geq 2$. The assertion follows by simple arithmetic.
2) It follows from (1) that $\mathrm{Con}(B_nD_{q+1}) \geq l - 1$. Since γ_{q+1} is a homotopy section of a map to X we find $\mathrm{Con}(B_nD_{q+1}) = l - 1$ and hence that $\mathrm{Con}(D_{q+1}) = l - 2$.
3) This is again Lemma 3.1. applied to $\sigma = B_n\eta_{D_rX}$ and $f = \gamma_r$.
4) The assertion follows from Lemma 4.1 (1) and Proposition 3.2 applied to $\alpha = \eta_{D_qX}$ and $\beta = \Omega\gamma_q$. □

For the following lemma all spaces are assumed to be of the homotopy type of a CW-complex. The proof is the same as that of Lemma 5.7 in Clapp and Puppe's paper [9] if one replaces the spaces J_X^nB there by B_nD, and hence is omitted.

Lemma 4.3 *Let Y be a space. If there is a map*

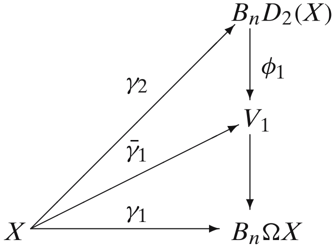
$$g : Y \to B_nD$$

where D is an $(l-2)$-connected A_n space (with $l \geq 2$) such that $Con(g) = l$ and

$$dim(Y) \leq k := Con(g) - 1 + nl.$$

Then we have $Cat(Y) \leq n$.

Proof of Theorem 1.1 In the case where $\mathrm{Cat}(X) = n$ is assumed then X carries a genuine $B_n\Omega$ coalgebra structure of order 1 by Corollary 2.6. Conversely, if γ_1 defines a genuine homotopy coalgebra of order 1 on X there is, by the universal property of the homotopy pullback V_1, a lift up to homotopy $\bar{\gamma}_1$ to V_1 in the following diagram.



By Proposition 4.2(4)

$$\mathrm{Con}(\phi_1) \geq 2(n+1)l - 5 \geq \dim(X),$$

so there is a lift γ_2 of $\bar{\gamma}_1$ to $B_nD_2(X)$. Hence X admits a weak homotopy coalgebra structure of order 2. By Proposition 4.2(1) we have

$$\mathrm{Con}(\gamma_2) \geq (2n+1)l - (n+3).$$

Now compute

$$\mathrm{Con}(\gamma_2) - 1 + nl \geq 2(n+1)l - 5 \geq \dim(X)$$

so that $\mathrm{Cat}(X) = n$ by Lemma 4.3. □

Proof of Theorem 1.2 By Proposition 4.2

$$\mathrm{Con}(\gamma_r) = (n+1)l + (r-1)(n(l-1)-1) - 2.$$

Now apply 4.3. □

We close with some open problems.

Problem 1 In contrast to the case $n = 1$, we do not know if any space X with $\mathrm{Cat}(X) = n$ admits a homotopy coalgebra structure of infinite order over $B_n\Omega$. The problem is due to the fact that spaces of the form B_nZ, Z an A_n space, are special in the class of all spaces with $\mathrm{Cat}(X) = n$, except if $n = 1$. Also, we do not know how to utilize Theorem 2.5 in order to lift a structure map from V_r to B_nD_{r+1}.

Problem 2 The conditions that show up in the definition of a coalgebra of order r can be seen as higher order forms of homotopy associativity. Notions of homotopy commutativity are incorporated in the treatment of [1] by use of the cotriple $\Sigma^n\Omega^n$. This is also possible in our case. For Cat one demands that the attached spaces in a cone decomposition are suspensions of higher degree than given by Cornea's theorem. For cat one imposes coherence conditions on the homotopies which contract the subsets of a covering. So one can ask the following question. Is there an E_n version of the results in this paper? We hope to come back to this point in a later paper.

Problem 3 How does the notion of a (weak) coalgebra translate into a condition on the structure map into the fat wedge?

Problem 4 How is a coalgebra structure expressed in terms of models in the rational category? Can one explain the result of Felix and Thomas that $\mathrm{cat}(X) = \mathrm{Cat}(X)$, for X rational, 1-connected, and of finite type with $\mathrm{cat}(X) = 2$, in these terms?

Problem 5 Are the estimates in Theorems 1.1 and 1.2 best possible?

Problem 6 Are there results similar to the ones in this paper for cocat?

Appendix

The main goal of this appendix is to prove Theorems 2.1 and A.3.4.

A.1 Review on A_n theory in the language of trees

This section is expository. We review basic definitions about A_n theory. It will be assumed that the reader has some familiarity with the language of categorical and homotopical algebra (see [7] or [35] for the former and [41] or [26] for the latter). Moreover, for basic facts on topological operads and theories, [6], [36] can be consulted.

A non Σ operad in topological spaces is a collection $P_{k\geq 0}$ of spaces which come with maps

$$\gamma : P_k \times P_{j_1} \times \cdots \times P_{j_k} \to P_{j_1+\cdots+j_k}$$

satisfying certain associativity and unity relations. An A_n space is a space X together with an action of a non Σ operad which is contractible up to level n. From now on we skip the prefix non Σ since symmetric operads will not emerge. Our operads P will be reduced, i.e. we assume that $P_0 = *$. We wish to be more specific about the choice of operad. The bar construction, invented by Boardman and Vogt, of an operad (or theory) Θ produces a cofibrant model $W\Theta$ of Θ in suitable model/cofibration categories of topological operads [5], [44], [55]. The based version which we will actually use is written W'' in [6]. Applied to the operad of associative monoids A which is the one point space at all levels it produces an operad which has as underlying spaces cubical subdivided associahedra. First, following Lawvevre, let us give:

Definition A. 1.1 A *monoid structure* on a space X is given by a family of maps

$$\lambda_i : X^i \to X \quad i = 0, 1, 2 \cdots$$

such that

i) $\lambda_1 = 1_X$
ii) $\lambda_n \circ (\lambda_{r_1} \times \ldots \times \lambda_{r_n}) = \lambda_m$, with $m = r_1 + \cdots + r - n$.

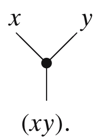
In case $(X, e) \in Top_*$ we assume that $\lambda_0 = e \hookrightarrow X$.

For example, if X is a topological monoid, define

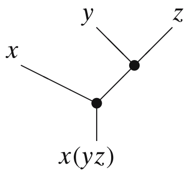
$$\lambda_i(x_1, \ldots, x_i) = x_1 \cdots x_i$$

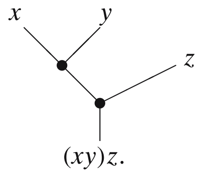
which is well defined because of the associativity. Thus a topological monoid is an algebra over the operad A.

A multiplication on an H-space X, $X^2 \to X, (x, y) \to xy$ is represented by a box with two inputs x, y and one output xy:

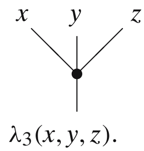


By wiring boxes together composites are obtained:

(1) 

(2) 

For a monoid X these two operations agree with λ_3 which we represent by the tree:

(3) 

If X is a homotopy associative H-space the operations 1 and 2 are only homotopic and this is accounted for by assigning a length between 0 and 1 to the internal edge between the two boxes. The two composite operations are represented by the planar trees (1), (2) with length $t = 1$. Letting t approach zero joins (1) to (2) through (3).

This picture generalizes to an arbitrary number of inputs as follows. Let T be a tree embedded in the plane which we orient down by gravity. The inputs (which are ordered since T is planar) have no beginning vertex and are called the **twigs**. All internal edges have lengths assigned to them. The output, called the **root**, has no end vertex with it. If we vary the length of all internal edges between zero and one, this set makes up a cube $C(T)$. The tree corresponding to λ_0 has no input and one output; it is called a **stump**. The trivial tree represents the identity operation.

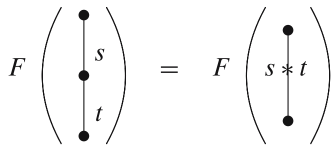
Definition A. 1.2 Let C_j denote the disjoint union of all $C(T)$ where T has $j \geq 1$ twigs and suppose $(X, e) \in Top_*$. A $W''A_{(n)}$*-structure* on X, $1 \leq n \leq \infty$, is defined to be a map

$$F : \coprod_{1=j}^{n} C_j \times X^j \to X$$

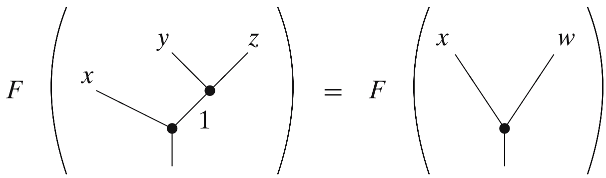
satifying the following relations. Let us represent an element from $C_j \times X^j$ by a tree with input $x_1 \cdots x_j$ to which F assignes a value of the output.

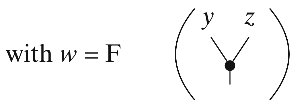
1) Any edge of length 0 may be removed, amalgamating the boxes at the ends to a single box.
2) Any box with only one input may be removed (unless it constitutes the whole tree). If the resulting edge is internal and input and output of the box are connections of length s and t, respectively, the edge after removal obtains length

$s * t = s + t - st$. Here we assume that the root and the twigs have the fictive length 1.

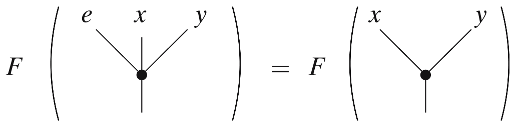


3) If T is a tree with a connection k of length 1 we may substitute the subtree D having k as output by the value of the output of D.

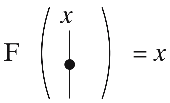




4) Let T be a tree different from the tree with input e and no internal edge. Then all parts of T with inputs e may be deleted i.e.



5)



Note that the basepoint e serves as a strict unit. This is to say, X is an algebra for the operad $W''A_{(n)}$, defined as follows.

Definition A. 1.3 Let $W''A_j$ be the quotient of C_j by the relations 1, 2 in A.1.2 and the relation which prunes away all stumps by shrinking their outgoing edges. The composition

$$W''A_j \times W''A_{r_1} \times \cdots \times W''A_{r_j} \to W''A_{r_1+\cdots+r_j}$$

is defined by grafting the tree from $W''A_{r_i}$ into the i^{th} input of the element from $W''A_j$ and giving the new internal edges length one. The operad generated by $W''A_j$ for $j \leq n$ will be denoted $W''A_{(n)}$. A space X with an algebra structure over $W''A_{(n)}$ is called an *A_n space*.

As is well known [6], the space $W''A_j$ is a cubical subdivision of the associahedron K_j and and the notion of A_n space just given coincides with that of Stasheff (see also [36]).

There is a map of operads

$$W^{''}A_{(\infty)} \to A$$

obtained by shrinking all edges. For a topological monoid X, the induced $W^{''}A_{(n)}$-structure will be called the **trivial** $W^{''}A_{(n)}$-structure.

A.2 A_n maps and partial monoids

The category which has A_n spaces as objects and A_n homomorphisms which are maps commuting with the action on the nose as morphisms forms a closed model category [45], [40]. Weak equivalences and fibrations are weak equivalences and Serre fibrations in Top_*. The class of cofibrations is made out of retracts of relative free extensions (see [45]). But for our needs it is too much to ask for commutativity on the nose. The description of the space of maps respecting two A_n structures up to homotopy and all coherencies needs the theory of bicoloured theories or operads [6]. The proper way to make this description may be in terms of a kind of model theory for n-categories. But the theory of n-categories, let alone the homotopy theory, is still in its infancy; see [33] for the state of the art.

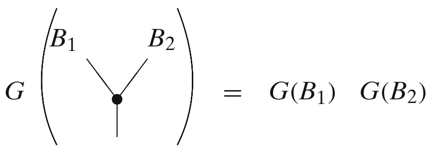
Fortunately, it is enough for our purposes to look at maps from an A_n space X to a monoid N with the trivial A_n structure. Reducing attention to this situation has some tradition [50], [51], [54].

Definition A. 2.1 Let $g : X \to M$ be a map from the A_n space (X, F) to the monoid M. An A_n *structure* on g is a map:

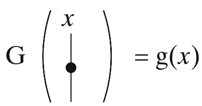
$$G : \coprod_{j=1}^{n} C_j \times X^j \to M$$

which satisfies

(A.2.1.1): (A.1.2.1) with F replaced by G.
(A.2.1.2): (A.1.2.2) with F replaced by G, and only applied when the output is not the root.
(A.2.1.3): (A.1.2.3) with F replaced by G except that the label w is still defined in terms of F.
(A.2.1.4): (A.1.2.4) with F replaced by G.
(A.2.1.5):



(A.2.1.6):



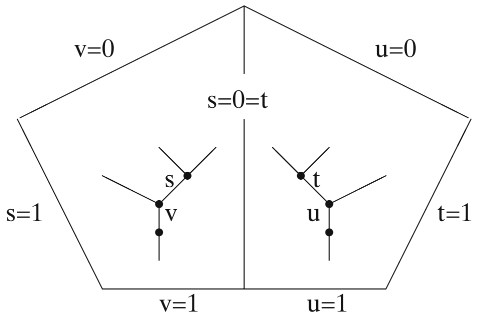
holds.

We set $M''A_j$ to be the quotient of C_j modulo the relations A.2.1.1, A.2.1.2 and the relation which shrinks all stumps. Then an A_n structure on g induces (and is the same thing) as an action map

$$G : \coprod_{0\le j\le n} M''A_j \times X^j \to M$$

which satisfies the relatons given above.

Remark Note that, because A.2.1.2 is not applied if the output is the root, $\dim(MA''_j) = \dim(W''A_j) + 1$. The space $M''A_j$ can be viewed as the union of all cubes $C(T)$, where T is a tree without stumps with j inputs such that the box at the root has exactly one and the other boxes exactly two inputs. These cubes $C(T)$ are glued along their lower edges, i.e., where the parameter corresponding to an edge is zero. Splitting off the parameter of the incoming edge of the root we find that $M''A_j \simeq W''A_j \times I$ where I is the unit interval. The picture below shows $M''A_3$.



A map admits an A_n structure if and only if it admits one with Stasheff's definition of an A_n map given in [36]. This fact appears already in [6] and we will use it without comment. Next, we recall the construction of functorial cofibrant models in the model category of A_n spaces and A_n homomorphisms. An A_n space (X, F) is a retract of a filtered monoid M. This construction goes back to Adams (unpublished, but see [51]); it appears in [6] and [54]. One sets

$$M = \coprod_{j=1}^{\infty} C_j \times X^j / \sim$$

with the relations A.2.1 except the last one and A.2.1.3 is only applied to trees with at most n twigs. The filtration M_j is defined by the number of inputs and multiplication by amalgamating the trees at the root. We note that the unit is represented by the tree with one box and input e. Consider the map $j : X \hookrightarrow M$ defined by

j(x) = 

It is an A_n map with structure given by the quotient map

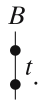
$$\coprod_{j=1}^{n} C_j \times X^j \to M.$$

There is a strong deformation retraction

$$p : M_n \to X$$

defined by $p(B) = F(B)$ for a representing tree B. The deformation is given by

$$h_t(B) =$$



Replacing X by M_n has the advantage that there is a partially defined monoid multiplication on the latter. To be more precise let us give the

Definition A. 2.2 *A partial monoid* M is a space with a subspace $M_{\langle 2\rangle} \subset M \times M$ and a map $M_{\langle 2\rangle} \to M$, written $(m, m') \to m \cdot m'$, such that:

i) There is an element $e \in M$ such that $m \cdot e$ and $e \cdot m$ are defined for all m in M, and $e \cdot m = m \cdot e = m$.
ii) $m \cdot (m' \cdot m'') = (m \cdot m') \cdot m''$ in the sense that if one side is defined then the other is too, and they are equal.

The notion of a partial monoid in relation to A_n theory was already studied by Husseini [27], and Segal applied it to the A_1 case of a pointed space [46]. We put a partial monoid structure on the space M_n considered above by setting

$$M_{\langle 2\rangle} = \bigcup_{i+j \le n} M_i \times M_j.$$

The multiplication is given by restriction of the monoid structure of M. There is an induced A_n space structure on M_n defined by sending $(\tau, m_1, \ldots, m_j)$ to $[\tau, F(m_1), \ldots, F(m_j)]$ for $\tau \in C_j$ and $j \le n$. With respect to this structure the map $p : M_n \to X$ is an A_n homomorphism. Similarly, there is a monoid structure and an A_∞ structure on M. Note that, due to relation A.1.2.3, the inclusion of M_n into M is an A_n homomorphism. In case X is a well pointed CW-complex, M_n is a cofibrant A_n algebra. A proof of this fact can be found in [44]. An A_n map g into a monoid N factorizes as $g = \tilde{g} \circ j$ in a unique way with a partial monoid map $\tilde{g} : M_n \to N$. Hence, for X a CW-complex, g factorizes over the cofibrant model M_n.

We need some more preparations.

Definition A. 2.3 A filtration of a pointed space (M, e) with

$$e = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_n = M$$

with a partial monoid structure such that

$$M_{\langle 2\rangle} = \bigcup_{i+j \le n} M_i \times M_j$$

is called an A_n *rectification* of M_1 if there is a strong deformation retraction

$$p : M_n \to M_1$$

with homotopy h_t which restricts to a strong deformation retraction on M_i for $i < n$.

The name (and the definition) is justified by the next proposition which is a variation of Boardman's proof of homotopy invariance in [5, Sects. 8.1,8.2,6.2]. The difference to Boardman's theorem lies in the fact that two different structures are compared. Since the multiplication on a partial monoid is in general only defined on a subset of the cartesian product a partial monoid is not a priori an A_n space. A similiar result appears in Husseini's paper [27, 4.5] in the context of his reduced product constructions with relations. For the proof the cubical subdivision of the associahedra K_i will be of much use.

Theorem A. 2.4 *A well pointed space X admits an A_n structure if and only if there is an A_n rectification with $X = M_1$.*

Proof The construction of M_n above gives the rectification from an A_n structure. So suppose X has an A_n rectification M_n with j, p and h as above. Note that although M_n is not an A_n space because the multiplication on M_n is not everywhere defined we can multiply n image elements of X inside M_n. So the notion of an A_n structure G below makes sense.

We prove the existence of an A_n structure F on X such that j carries an A_n map structure G by a double induction on the index (k, i) in $(W'' A_i)^k$, where $T \in (W'' A_i)^k$, for a representing tree T, if the number of internal edges is $\leq k$ and i is the number of twigs. By abuse of notation, we write $C(T)$ for the image of the cube $C(T)$ under the identification map to $(W'' A_i)^k$. Moreover, for G and a representing tree T' in $M'' A_i$ with $k+1$ internal edges we split off the parameter of the incoming edge to the root and write $G(T, t)$ for $G(T')$ (see the remark after A.2.1.). Note that, due to relation A.1.2, we have $W'' A_i = (W'' A_i)^{i-2}$ for $i > 1$.

For T with $k = 0$ and all $i \leq n$ we define

$$F(T, x_1, \ldots, x_i) = p(x_1 \cdots x_i)$$

and

$$G(T, t, x_1, \ldots, x_i) = h_t(T, x_1, \ldots, x_i).$$

Here we identify each x with its image under $j : X \to M_n$.

Suppose that $F(T, -)$ and $G(T, t, -)$ are found for all T with index (k', i) with $k' < k$ and all $i \leq n$ such that the conditions listed in A.1.2 and A.2.1. hold and $G(T, 1, -) = jF(T, -)$. Let T be a tree with k internal edges. Define $Q(T)$ to be the subset of $C(T)$ given by the trees for which A.1.2.1, A.1.2.2 or A.1.2.3 holds. Then $Q(T) \subset C(T)$ is a closed cofibration. Because X is well pointed, the inclusion of the fat wedge $X^{<i>}$ into X^i is a closed cofibration. Finally, by the product theorem in [52], the inclusion of the pair

$$C(T) \times X^{<i>} \cup Q(T) \times X^i \subset C(T) \times X^i$$

is a closed cofibration also. Now F and G are already found on

$$C(T) \times X^{<i>} \cup Q(T) \times X^i$$

and $G(T, 0, -)$ is also known. By the homotopy extension lifting property of homotopy equivalences (HELP for short see [6, Appendix Theorem 3.5]), applied to $j : X \simeq M_i$ and to the map from $C(T) \times X^i$ to M_i which sends $(T, x_1, \ldots, x_i)$ to $x_1 \cdots x_i$, F and G can be extended to $C(T) \times X^i$ and to $C(T) \times I \times X^i$ respectively. These extensions patch together and define extensions to $(W'' A_i)^k \times X^i$ and $(W'' A_i)^k \times I \times X^i$ respectively. Thus, the inductive assumption is true for (k, i) and this holds for all $i \leq n$. □

Proof of Theorem 2.1 We show that there is an A_n rectification for P. This suffices by A.2.4. Consider the factorization of g into

$$X \xrightarrow{j} M_n \xrightarrow{\tilde{g}} N$$

with a map of partial monoids $\tilde{g}$ where M_n is the rectification of X constructed above. Consider the homotopy pullback:

$$\begin{array}{ccc} M_n(P) & \longrightarrow & K \\ \downarrow & & \downarrow f \\ M_n & \xrightarrow{\tilde{g}} & N. \end{array}$$

Define a filtration on the space $M_n(P)$ by pulling back along the restriction of $\tilde{g}$ to M_i for $i \leq n$. The i^{th} filtration may be given explicitly as:

$$M_i(P) = \{m_i, w, k | m_i \in M_i, w : I \to N, w(0) = \tilde{g}(m_i), w(1) = f(k), k \in K\}.$$

Pointwise multiplication of paths and the multiplications on M_n and N give us a partial monoid structure on this space. This structure satisfies

$$M(P)_{\langle 2\rangle} = \bigcup_{i+j\leq n} M_i(P) \times M_j(P)$$

because the corresponding equality holds for M_n. A strong deformation retraction

$$\bar{p} : M_n(P) \to P$$

is defined as follows. First note that, for every element m from M_n, the length $s(m)$ of the ingoing edge to the root assignes a number between zero and one to m. Moreover, the homotopy h_t which starts at the identity and ends at the retraction p deforms this length from $s(m)$ to 1. Define

$$\bar{p}(m, w, k) = (p(m), \tilde{w}_m, k)$$

with

$$\widetilde{w}_m(t) = \begin{cases} \widetilde{g}(h_{(1-\frac{t}{1-s(m)})}(m)) & \text{for } t \leq 1 - s(m) \\ w(\frac{t-(1-s(m))}{s(m)}) & \text{for } t \geq 1 - s(m) \end{cases}$$

if $0 < s(m) < 1$, and

$$\tilde{w}_m(t) = \tilde{g}(h_{(1-t)}(m)), \quad \tilde{w}_m(t) = w(t)$$

if $s(m) = 0, s(m) = 1$, respectively. The homotopy $\bar{h}_t$ is given as

$$\bar{h}_t(m, w, k) = (h_t(m), \tilde{w}_{h_{(1-t)}(m)}, k).$$

Because $h_t(M_i) \subset M_i$ the homotopy $\bar{h}_t(-)$ leaves $M_i(P)$ invariant. The induced maps from $M_n(P)$ to M_n and K are both maps of partial monoids. So they induce maps on classifying spaces. □

A.3 Adapting the standard theory of classifying spaces to partial monoids

The construction of classiying spaces for partial monoids which we describe generalizes Milgram's construction for monoids [37], see also [53]. Let M be a partial monoid and let $M_{\langle n\rangle} \subset M^n$ be the subset of all elements $(m_0,\ldots,m_{n-1})$ such that the product $m_0\cdots m_{n-1}$ and all products which can be obtained by leaving out some of the m_i are defined. Define a simplicial space $E_{(*)}$ by $E_{(n)} = M_{\langle n+1\rangle}$. The face and degeneracy operators on $E_{(n)}$ are given by the multiplication and insertion of the unit respectively as

$$\begin{aligned}
\partial_i(m_0,\ldots,m_n) &= (m_0,\ldots,m_i m_{i+1},\ldots,m_n) \quad i<n\\
\partial_n(m_0,\ldots,m_n) &= (m_0,\ldots,m_{n-1})\\
s_i(m_0,\ldots,m_n) &= (m_0,\ldots,m_i,e,m_{i+1},\ldots,m_n).
\end{aligned}$$

This is a modification of the nerve construction [47] which is necessary because one cannot view M with its multiplication as a category in the usual way since the multiplication of two elements in M is not always defined.Let $B_{(n)}$ be the simplicial space with $B_{(n)} = M_{\langle n\rangle}$. The simplicial operators on $B_{(*)}$ are given as

$$\begin{aligned}
\partial_0(m_1,\ldots,m_n) &= (m_2,\ldots,m_n)\\
\partial_i(m_1,\ldots,m_n) &= (m_1,\ldots,m_i m_{i+1},\ldots,m_n) \quad 0<i<n\\
\partial_n(m_1,\ldots,m_n) &= (m_1,\ldots,m_{n-1})\\
s_i(m_1,\ldots,m_n) &= (m_1,\ldots,m_i,e,m_{i+1},\ldots,m_n).
\end{aligned}$$

The projection onto the last n factors $M_{\langle n+1\rangle} \to M_{\langle n\rangle}$ defines a map of simplicial spaces

$$p : E_{(*)} \to B_{(*)}.$$

The spaces E and B are now defined as $|E_{(*)}|$,$|B_{(*)}|$, the geometric realizations [47] of $E_{(*)}$ and $B_{(*)}$. We write $|p| : E \to B$ for the map induced by p. Let E_k, B_k denote the filtrations of E, B given by the skeleta. A contraction of E_k in E_{k+1} is defined by (see [6, pp.175–177])

$$H_t(m_0,\ldots,m_k,t_1,\ldots,t_k) = (e,m_0,\ldots,m_k,t,t*t_1,\ldots,t*\cdots*t_k).$$

Here the model of the n-simplex used is

$$\Delta_n = \{(t_1,\ldots,t_n) \in R^n | 0 \le t_1 \le \cdots \le t_n \le 1\}.$$

Let D_k denote the realization of the subspace of $E_{\le k}$ which has as k-simplices all $(e,m_1,\ldots,m_k)$. The space $D_k \subset E_k$ is the image of the homotopy H and is clearly contractible.

Remark Alternatively, as in in [37] and [53], one could define the space B as the orbit space of a partial M action as follows. The simplicial space $E_{(*)}$ carries a partially defined M action

$$\begin{aligned}
M_{\langle n+2\rangle} &\to M_{\langle n+1\rangle} = E_n\\
((m_0,m_1\ldots,m_{n+1})) &\to (m_0m_1,\ldots,m_{n+1}).
\end{aligned}$$

Because $(m_0,\ldots,m_n) = m_0(e,\ldots,m_n)$, each point lies in exactly one maximal partial orbit of the form $M(e,\ldots,m_n)$. Define a simplicial space $B_{(*)}$ as the quotient of $E_{(*)}$

under the equivalence relation corresponding to this partition. Note that the quotient map p defines a map $|p| : E \to B$. It follows from the fact that geometric realization commutes with products that there is an induced partial action of M on E and we still have $E/M = B$. It can be shown that the two constructions are homeomorphic in the case that M is the partial monoid corresponding to an A_n space. In general the topology may be different since the maximal orbits need not be closed. In order to avoid these matters we took, in line with [50] and [6, pp. 175–177], the route above. For an action of a group G all orbits are a copy of G. If we weaken the assumptions on M the orbit structure becomes more complicated. In case M is a monoid at least the maximal orbits of E are homeomorphic to M [53, p. 354]. This does not hold for a partial monoid in general, but there is still a homotopy equivalence in many cases of interest.

In order to produce the right homotopy type we have to impose cofibration conditions on the simplicial spaces. Recall that a pair (X, A) of Hausdorff spaces is called a **NDR pair** if the inclusion of A in X is a closed cofibration.

A simplicial space $X_{(*)}$ is called **proper** if all degeneracies are closed cofibrations. We call a filtration of a space X proper if the pairs (X_{i+1}, X_i) are NDR pairs.

Lemma A. 3.1 *Let $(X, *)$ be a well pointed A_n space with X Hausdorf and M_n the rectification defined above in A.2. Then*

$$e \subset M_1 \subset \cdots \subset M_n$$

is a proper filtration.

Proof According to [54, p. 324], the space M_i is obtained from M_{i-1} by attaching $Q_i \times X^i$ along a space $R = DQ_i \times X^i \cup Q_i \times X^{<i>}$ where $I \times Q_i^{'} \cong Q_i$, $Q_i^{'}$ is homeomorphic to Stasheff's $K_i \cong I^{i-2}$ and $DQ_i \subset Q_i$ is a closed cofibration. Because X is assumed well pointed, it follows from Steenrod's results in [52, Sects. 6.3, 8.5] that first $X^{<i>} \subset X^i$ and then $M_{n-1} \subset M_n$ is a closed cofibration. Since $(X, *)$ is identified with (M_1, e) the assertion follows. □

Lemma A. 3.2 *For $(X, *)$ and M_n as in Lemma 4.1 the simplicial spaces $E_{(*)}, B_{(*)}$ are proper.*

Proof We show that

$$M_{\langle s\rangle} \times 0 \cup M_{\langle s-1\rangle} \times I \subset M_{\langle s\rangle} \times I$$

admits a retraction. Here $M_{\langle s-1\rangle}$ maps to $M_{\langle s\rangle}$ via s_i. It is well known that this is equivalent to s_i being a closed cofibration. Let us look at the products whose union makes up $M_{\langle s-1\rangle}$ and $M_{\langle s\rangle}$. It is

$$\bigcup M_{j_1} \times \cdots \times M_{j_i} \times \cdots \times M_{j_s} = M_{\langle s\rangle}$$

where the union is over all indices with $\sum_t j_t \leq n$. It follows from Lemma A.3.1 that all the inclusion maps

$$M_{j_1} \times \cdots \times M_0 \times \cdots \times M_{j_s} \to M_{j_1} \times \cdots \times M_{j_i} \times \cdots \times M_{j_s},$$

with $M_0 = e$, are closed cofibratoins. The corresponding retraction is given by the retraction for $M_{j_i} \times 0 \cup e \times I \subset M_{j_i} \times I$ times the identity on the other factors. These retractions patch together to define a retraction on $M_{\langle s\rangle} \times I$. This proves the lemma.

□

Proposition A. 3.3 *Let* $(X,*)$ *and* M_n *be as above. Then for* $m < n$ *the map* $|p| : E_m \to B_m$ *has homotopy fibre homotopy equivalent to* X. *In case* $m \geq n$, E_m *is contractible.*

Proof First note that for each simplicial level $m < n$ the map p_m has homotopy fibre $F_m \simeq X$. This is so because p_m for $m < n$ corresponds under the homotopy equivalence $X^m \simeq M_{\langle m \rangle}$ to the projection. If we replace $|p|$ by the fat realization [48, Appendix] $||p||$ the assertion follows from in [39, Lemmas 2, 3.] as in the proof of the main theorem in l.c. But due to Lemma A.3.2. both realizations are homotopy equivalent (see [48] or [15]).

Suppose now that $m \geq n$. By definition of M_n, if $(m_0, \ldots, m_m)$ are composable elements in M_n^{m+1} then for at least one i it is $m_i = e$. Let i_0 be the smallest such index. If $i_0 > 0$ then by the identifications which define E_m each element represented by $(m_0, \ldots, m_m, t_1, \ldots, t_m)$ is in E_{m-1}. In case $i_0 = 0$ we have $(m_0 = e, \ldots, m_m, t_1, \ldots, t_m) = H_{t_1}(m_1, \ldots, m_j, u_2, \ldots, u_j)$ with $u_s = \frac{t_s - t_{s-1}}{1 - t_{s-1}}$ with the convention that $\frac{0}{0} = 1$. Hence $E_m = D_m$. Since D_m is contractible the proposition is now proved. □

Remark Note that Puppe's results give us a homotopy equivalence, not only a weak equivalence, in Proposition 4.3, and that we did not make the assumption that X is a CW-complex. The results on quasi-fibrations in [17] can be used, as is traditionally done, to produce a weak homotopy equivalence from X to the homotopy fibre of $|p|$.

Recall that a covering $\{V_j | j \in J\}$ of a space X is called **numerable** if there is a partition of unity $\{u_j | j \in J\}$ such that $u_j^{-1}(0,1] \subset V_j$ holds. The covering is called **contractible** if the inclusion of V_j into X is homotopic to zero. A space X is called numerable contractible if X admits a numerable contractible covering. Paracompact Hausdorff spaces are characterized by the fact that every open covering is numerable. Since CW-complexes are paracompact they are numerable and in fact numerable contractible. It is well known since Puppe's proof of James theorem (see [16]) that many results about CW-complexes can be generalized for numerable contractible spaces.

Proposition A. 3.4 *Let* $(X,*)$ *be a well pointed, path connected, Hausdorf, and numerable contractible* A_n *space and* M_n *the rectification as above. Then there are homotopy equivalences (of pairs) which are natural with respect to maps which induce maps of partial monoids between the rectifictions:*

a.) $(D_i, E_{i-1}) \simeq (C(E_{i-1}), E_{i-1})$
b.) $B_i \simeq B_{i-1} \cup_{p_{i-1}} C(E_{i-1}) \quad i \leq n$
c.) $(E_i, E_{i-1}) \simeq (X^{*i+1}, X^{*i}) \quad i < n.$

Proof (a) If we replace $|\quad|$ by $||\quad||$ in the definition of D_i then there is a homeomorphism of the two pairs. This follows by inspection of the identifications in the definition of $||\quad||$. Because all degeneracies are closed cofibrations the fat realization functor and the realization functor are homotopy equivalent [47].

(b) First, note that the projection $|p|$ restricts to a relative homeomorphism from (D_i, E_{i-1}) to (B_i, B_{i-1}). Hence, we have

$$B_i \cong B_{i-1} \cup_{|p_{i-1}|} D_i$$

where $|p_{i-1}|$ is the restriction of $|p|$ to E_{i-1}. The assertion follows now from (a).

(c) We do an induction on i. For $i = 0$ the assertion holds because $E_0 = M_n \simeq X$. Since $i+1 \leq n$ the homotopy equivalences $X \simeq M_j$ induce a natural homotopy equivalence $X^{i+1} \simeq M_{\langle i+1 \rangle}$. From this and because we may ignore the identifications related to the degeneracies, by the same argument as above, we have homotopy equivalences induced by the inclusion

$$(E_i, E_{i-1}) \simeq (X^{i+1} \times \Delta_i \cup_\partial E_{i-1}, E_{i-1})$$

and

$$(D_i, E_{i-1}) \simeq (C(E_{i-1}), E_{i-1}) \simeq (e \times X^i \times \Delta_i \cup_\partial E_{i-1}, E_{i-1})$$

where ∂ denotes the restriction of the boundary operators. Since X is path connected so is M_i for all i. Consequently, left multiplication by $m \in M_i$, $l_m : M_j \to M_{i+j}$ is a homotopy equivalence. By [6, Appendix 4.12] M_i is numerable contractible because X is. An application of Theorem 9.3 in [16] shows that the map $M_i \times M_j \to M_i \times M_{i+j}$ which sends m, m' to m, mm' is a homotopy equivalence over M_i. Hence we do not change the homotopy type if we change the identification related to

$$\partial_0(m_o, m_1, \ldots, m_i, 0, \ldots, t_i) = (m_o m_1, \ldots, m_i, t_2, \ldots, t_i)$$

in $(X^{i+1} \times \Delta_i \cup_\partial E_{i-1}, E_{i-1})$ to

$$\partial_0^{'}(m_o, m_1, \ldots, m_i, 0, \ldots, t_i) = (m_1, \ldots, m_i, t_2, \ldots, t_i).$$

But $X * Y$ has the homotopy type of $X \times C(Y) \cup_{q_2} Y$ where q_2 is the projection onto the second factor of $X \times Y$. Let us write $\partial^{'}$ for the boundary operators with ∂_0 modified. Then there are homotopy equivalences

$$\begin{aligned}(X^{i+1} \times \Delta_i \cup_{\partial'} E_{i-1}, E_{i-1}) &\simeq (X \times C(E_{i-1}) \cup_{q_2} E_{i-1}, E_{i-1}) \\ &\simeq (X * E_{i-1}, E_{i-1}) \simeq (X^{*i+1}, X^{*i}).\end{aligned}$$

By inspection, the homotopy equivalences above are natural with respect to maps between A_n spaces which induce maps of partial monoids between the rectifications. In particular, A_n maps from an A_n space to a monoid have this property. □

Remark The proposition above shows that the homotopy type of the classifying space of the partial monoid M_n are the same as the ones of Stasheff's construction in [50] applied directly to X. Our proof follows closely Stasheff's arguments and we do not claim any originality. It may be possible to reach the conclusions of Lemma A.3.4 by a direct comparison with Stasheff's construction. It follows directly from the definitions that a map of partial monoids induces a map between classifying spaces.

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